Rate-Distortion function for gamma type source under absolute log distortion measure

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to appear in IEEE IT

@ Rate-Distortion function

$$R(D) = \inf_{q(v|u)} I(U;V),$$

subj. to

$$E_{p(u)q(v|u)}[d(u,v)] \le D$$

$$I(U;V) = \iint q(v|u)p(u)\log\frac{q(v|u)}{q(v)}\,du\,dv$$

$d(u,v)$: distortion measure $\quad d(u,v) \ge 0$

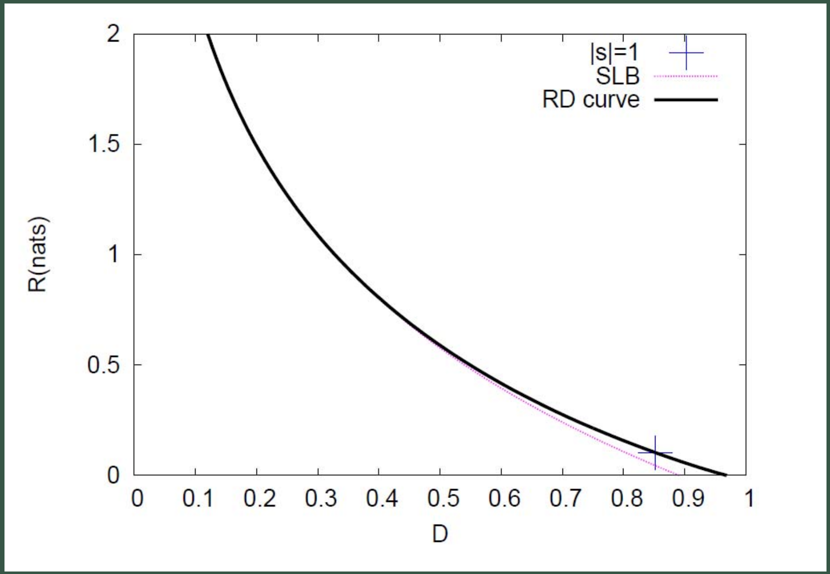
$p(u)$: source distribution

$$q(v) = \int p(u)q(v|u)\,du$$

Optimal Channel

$$q_s(v \mid u) = q_s(v) \cdot \lambda_s(u)\, e^{(u \quad v)} \qquad (s \leq 0)$$

marginal distribution of v



Rate-Distortion function

Known Results

$q_s(v)$: marginal

TABLE I

KNOWN RESULTS ON RATE-DISTORTION FUNCTIONS FOR CONTINUOUS SOURCES AND THEIR OPTIMAL RECONSTRUCTION DISTRIBUTIONS.[1]

Distortion $d(u,v)$	Source	Density $p(u)$	Reconstruction	SLB	References and notes
Absolute: $\|u-v\|$	Laplace	$\frac{\alpha}{2}e^{-\alpha\|u\|}$	discrete + Laplace	$=$	[8, p. 95, Ex. 4.3.2.1]: [13] deals with one-sided exponential.
	Gauss	$\frac{1}{\sqrt{2\pi\sigma^2}}\exp(-\frac{u^2}{2\sigma^2})$	discrete + continuous	$>$	[10]: generalized, e.g. to $p(u) \propto \exp(-\|u\|^\nu)$ $(1<\nu)$.
	Uniform	$1/2c$ $(-c<u<c)$	discrete + continuous	$>$	[11]: generalized to densities with constrained support.
	Gamma (log)	$\frac{\exp(\alpha u - e^u)}{\Gamma(\alpha)}$	discrete + continuous	$>$	[This paper]: generalized to densities with a heavy tail on one side.
	Squared Cauchy	$\frac{2}{\pi}(1+u^2)^{-2}$	continuous / unknown	$=$ / $>$	[8, p. 95, Ex. 4.3.2.2]: $R(D) = \text{SLB}$ for $D \le \sqrt{6}/5$.
	General	$p(u) \in \mathcal{P}_1$	$p(v) - D^2 p''(v)$	$=$	[8, p. 95]: $\mathcal{P}_1 = \{p(u); p(u) - D^2 p''(u) \ge 0, \forall u \in \mathbf{R}^1\}$.
Squared: $\|\|u-v\|\|^2$	Gauss	$\frac{1}{\sqrt{2\pi\sigma^2}}\exp(-\frac{u^2}{2\sigma^2})$	Gauss	$=$	[8, p. 99, Thm. 4.3.2], [7, p. 344, Thm. 13.3.2]
	General (on $\mathbf{R}^1$)	$p(u)$ (other than Gauss)	discrete	$>$	[9]
	Uniform (on $\mathbf{S}^1$)	$1/2\pi$ (on $\mathbf{S}^1$)	Uniform (on $\mathbf{S}^1$)	$=$	[14]: $d(x,y) = \|\|u-v\|\|^2 = 2 - 2\cos(\angle(u,v))$.
Itakura-Saito: $v-u+e^{u-v}-1$	Gamma (log)	$\frac{\exp(\alpha u - e^u)}{\Gamma(\alpha)}$	Beta (log)	$=$	[6]: $d(x,y) = \frac{x}{y} - \log\frac{x}{y} - 1$ $(u = \log x, v = \log y)$.

[1] The column SLB indicates if $R(D) = \text{SLB}$ or $R(D) > \text{SLB}$.

@ Our result

$$f(x) = \frac{\theta^{-\alpha} x^{\alpha-1}}{\Gamma(\alpha)} e^{-\frac{x}{\theta}}$$

$S = \log \frac{1}{2}$

$$d(x.y) = |\log x - \log y|$$

$$\Downarrow$$

$S = -0.8$

$$p(u) = \frac{1}{\Gamma(\alpha)} \exp(u\alpha - e^u)$$

$$d(u.v) = |u - v|$$

$S = -2$

$q_s(v)$

$\frac{1}{\lambda_s(u)}$

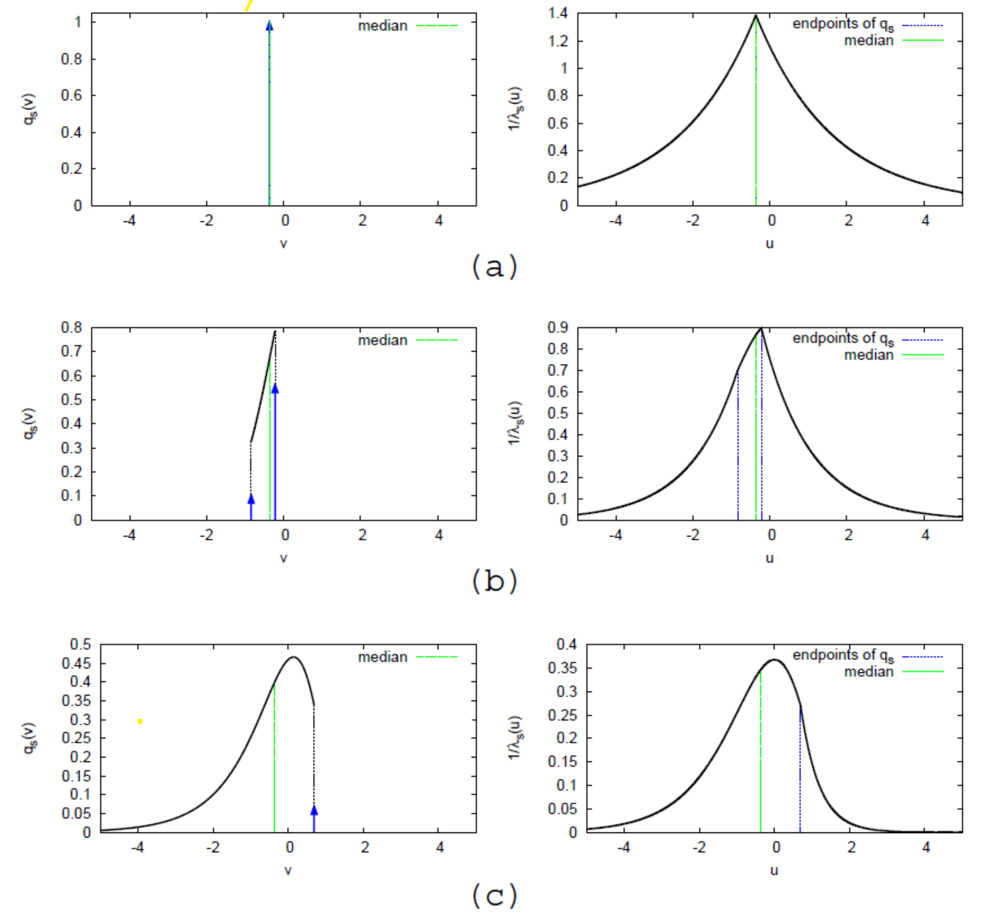


Fig. 1. Reconstruction density $q_s(v)$ (left panels) and function $\lambda_s(u)^{-1}$ (right panels) for $\alpha = 1$ and (a) $s = s_{\max} = -\log 2$, (b) $s = -0.8$ and (c) $s = -2.0$. Discrete components in $q_s(v)$ are represented by arrows whose length is equal to the coefficient in (27). Also indicated are the endpoints, $v^* - a_s$ and $v^* + b_s$, of $q_s(v)$ (in the right panels) and the median v^*.

Cosmological Parameters and Fisher Information Matrix

17 June 2016

Shiro Ikeda

Astronomy

- Different types of Data
- Electronic Magnetic waves. (different wavelengths)
- Measurement Technology ⟶ Big-Data
- Astro + Statistics (U.S, UK, Europe)
- Model (Physics) & Noisy data (Poisson, Gaussian)

- Astro-statistics
  - Better measurement with Statistics

    EHT, Compton camera
  - Keep their scientific methods with Big-Data

    HSC project
  - New method from statistics ?

- Astro-statistics

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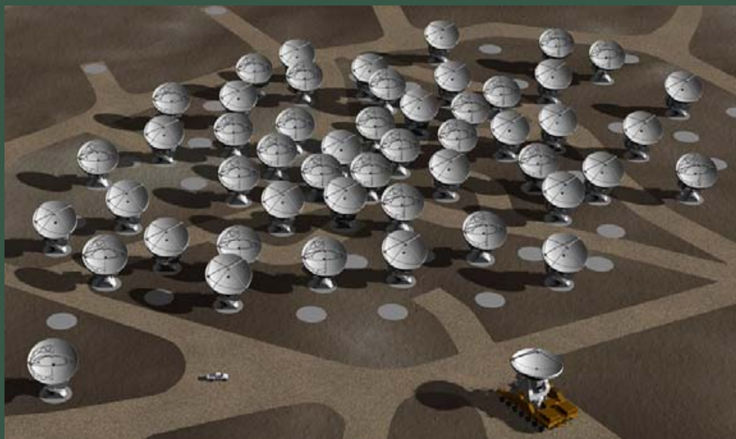
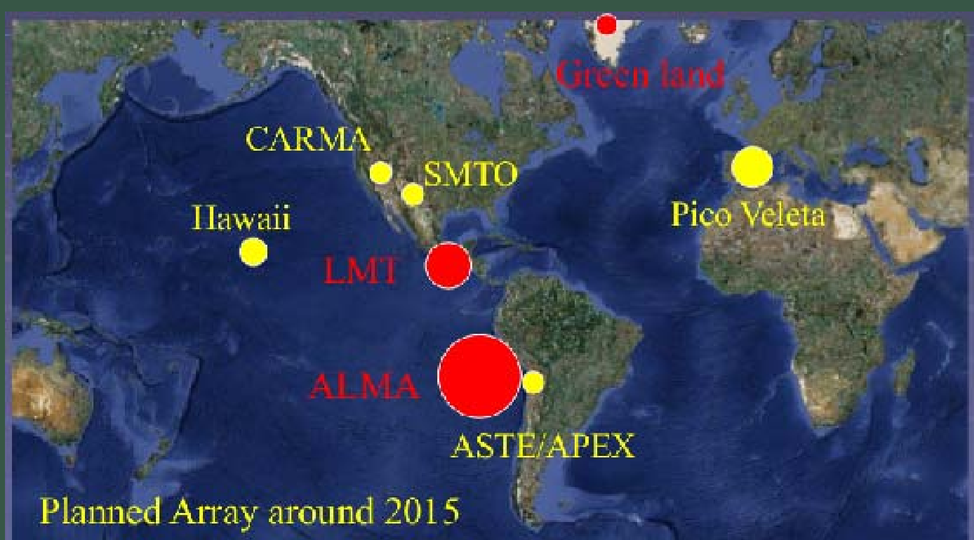
Event Horizon Telescope (cm – mm)



Honma, Tazaki, Hada (NAOJ), Akiyama (MIT)
Ikeda (ISM)

Honma, Akiyama, Uemura, Ikeda, PASJ, 66(5), 95, 2014
Ikeda, Tazaki, Akiyama, Hada, Honma, PASJ, 68(3), 45, 2016

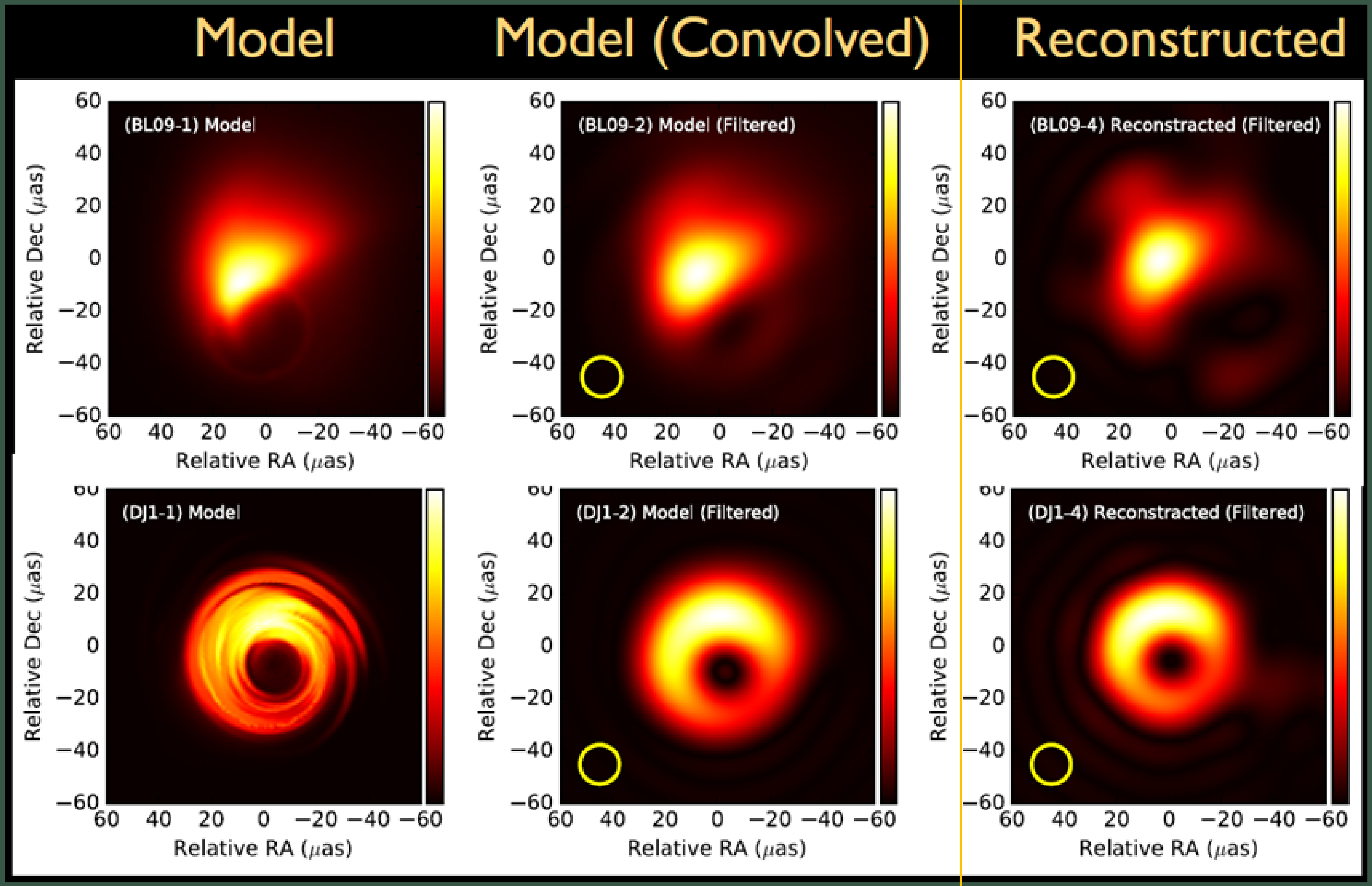
- Taking the image of a Black Hole.



- Image of Black Hole Shadow with Very Long Baseline Interferometer.
  - Imaging is an ill-posed problem.

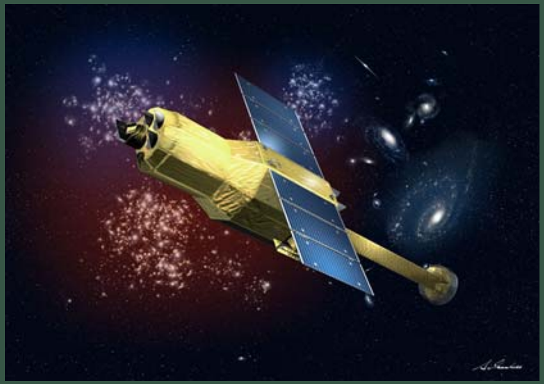
◎ EHT imaging

Proposed Method



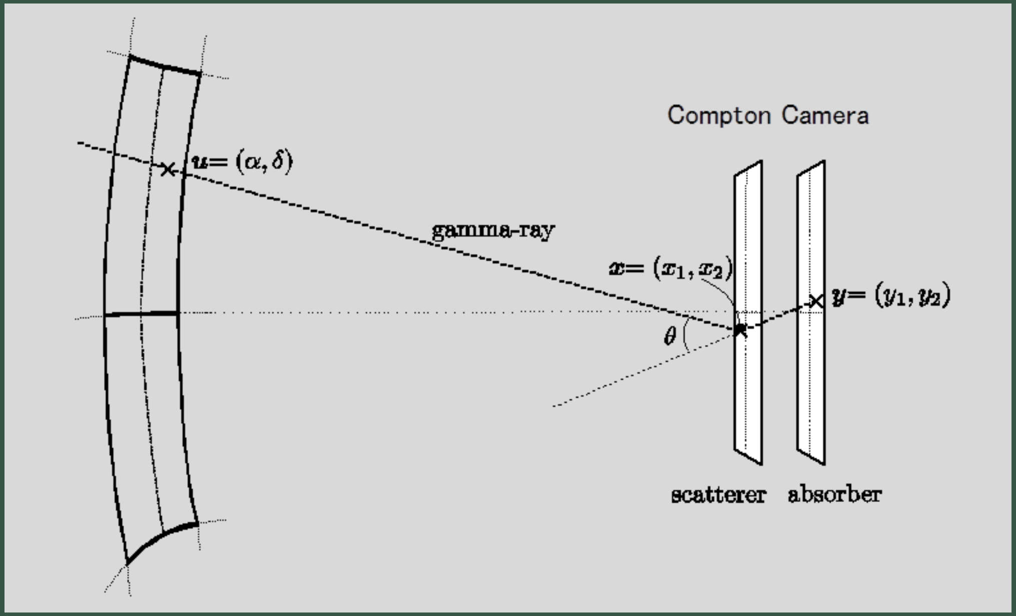
Ikeda, el.al, PASJ, 2016

Compton Camera Imaging (γ- ray)



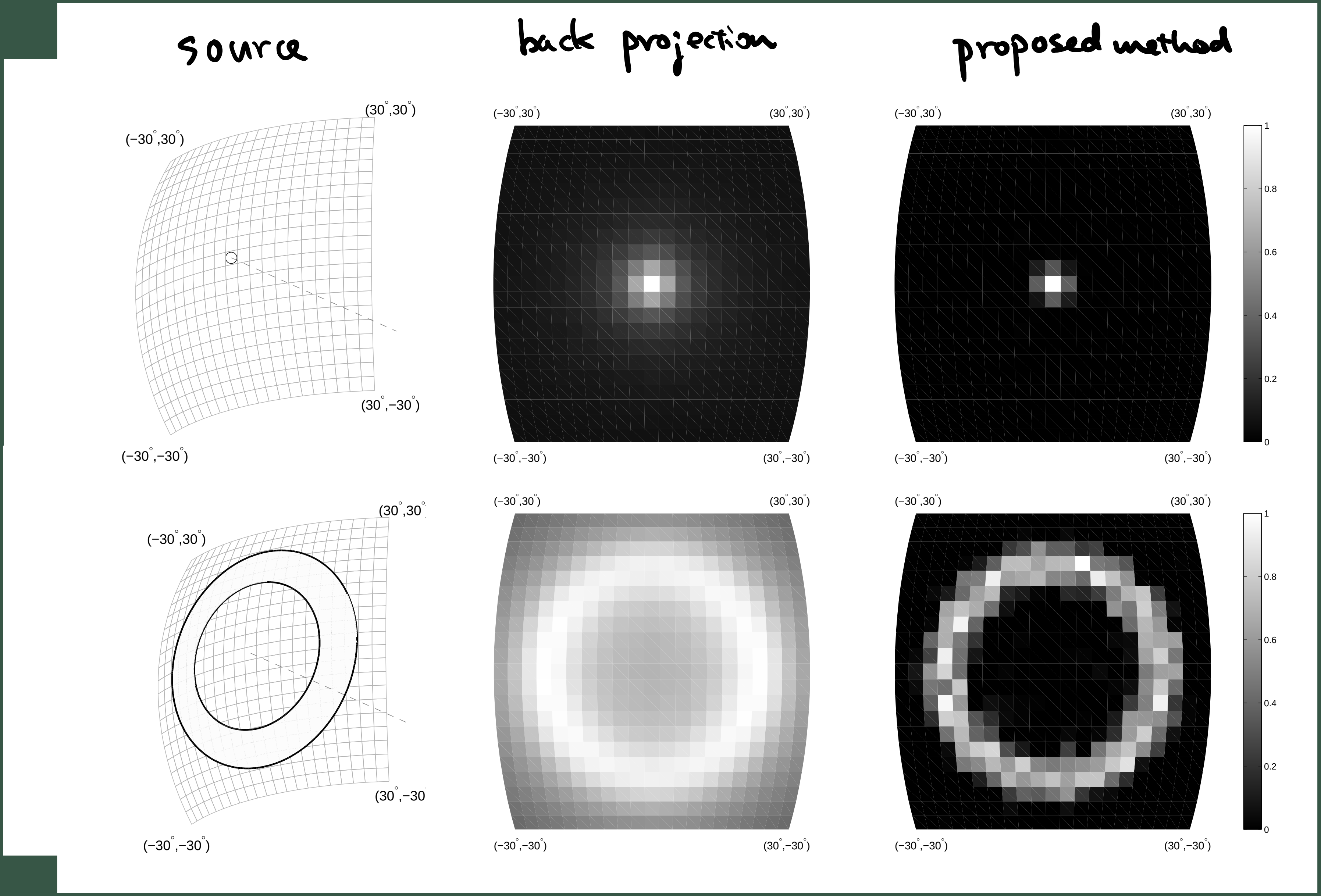
Odaka, Takahashi, et.al (JAXA), Ikeda (ISM)
Uemura (Hiroshima Univ)

Ikeda, Odaka, Uemura, Takahashi, Watanabe, Takeda, NIMA A, 760, 2014



Compton Scattering

Imaging of Gamma ray sources



- Astro-statistics
  - Better measurement with Statistics

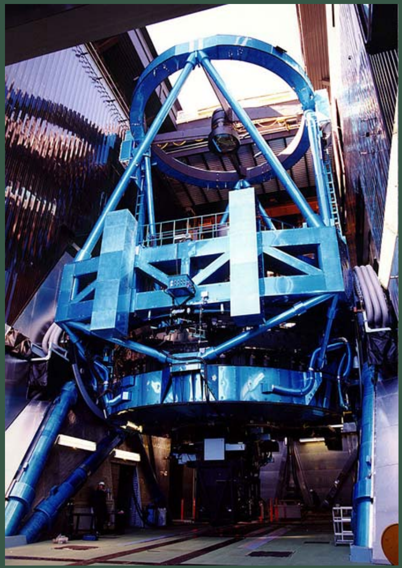
    EHT, Compton camera
  - Keep their scientific methods with Big-Data

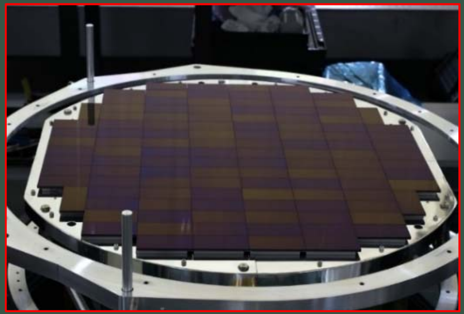
    HSC project
  - New method from statistics ?

@ HSC Project (visible light)

Yoshida, Tanaka, et. al (IPMU, U. of Tokyo)
Kawashima, et. al (Tsukuba Univ)
Ikeda, Morii, Iba, Koyama (ISM), Ueda, et. al (NTT)

- Installing a large CCD (HSC: Hyper-Suprime Cam) to one of the largest telescopes, IPMU and NAOJ started 5-year survey (2014 – 2019), spending 300 nights.
- 0.5 – 1 PB data will be delivered, big astronomical data.
- Goal is to determination of the cosmological parameters.





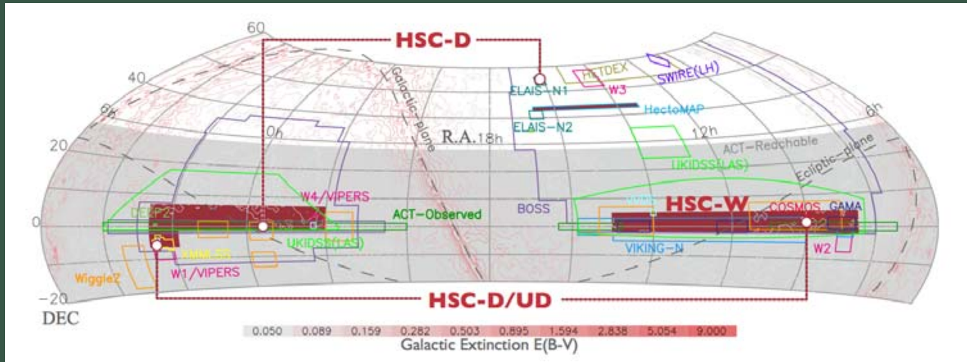


Figure 11: The location of the HSC-Wide, Deep (D) and Ultradeep (UD) fields on the sky in equatorial coordinates. A variety of external data sets and the Galactic dust extinction are also shown. The shaded region is the region accessible from the CMB polarization experiment, ACTPol, in Chile.

http://www.naoj.org/Projects/HSC/img/fields1.png

@ Cosmological Parameters.

6 parameters

$(\Omega_b h^2, \Omega_c h^2, \Omega_{dm}, n_s, A_s, \omega)$

[1] Parameter	2015F(CHM) (Plik)
$100\theta_{MC}$	1.04086 ± 0.00048
$\Omega_b h^2$	0.02222 ± 0.00023
$\Omega_c h^2$	0.1199 ± 0.0022
H_0	67.26 ± 0.98
n_s	0.9652 ± 0.0062
Ω_m	0.316 ± 0.014
σ_8	0.830 ± 0.015
τ	0.078 ± 0.019
$10^9 A_s e^{-2\tau}$	1.881 ± 0.014

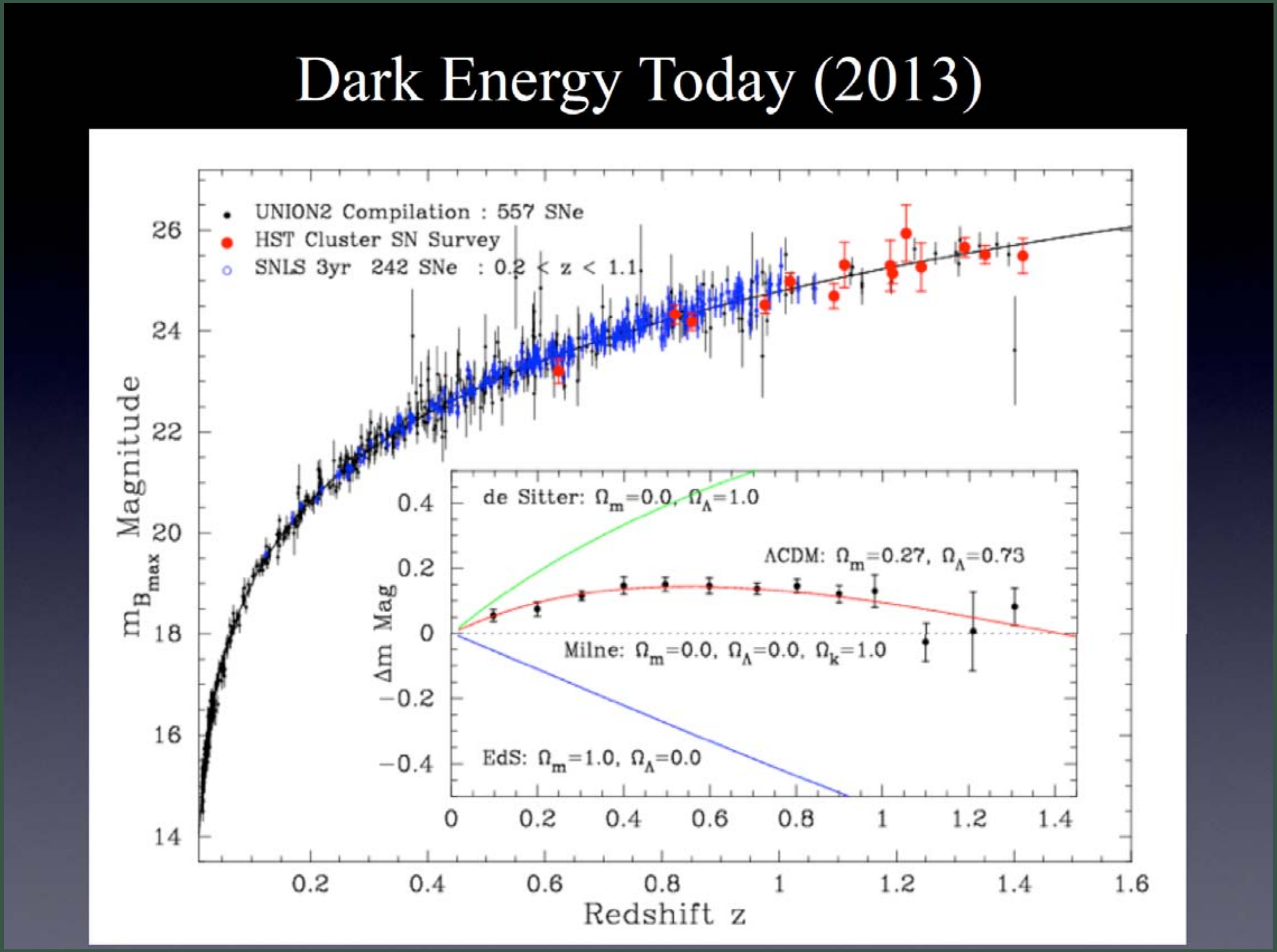
Estimate 6 parameters from observations

$\longrightarrow$ reduce the error bars. estimation

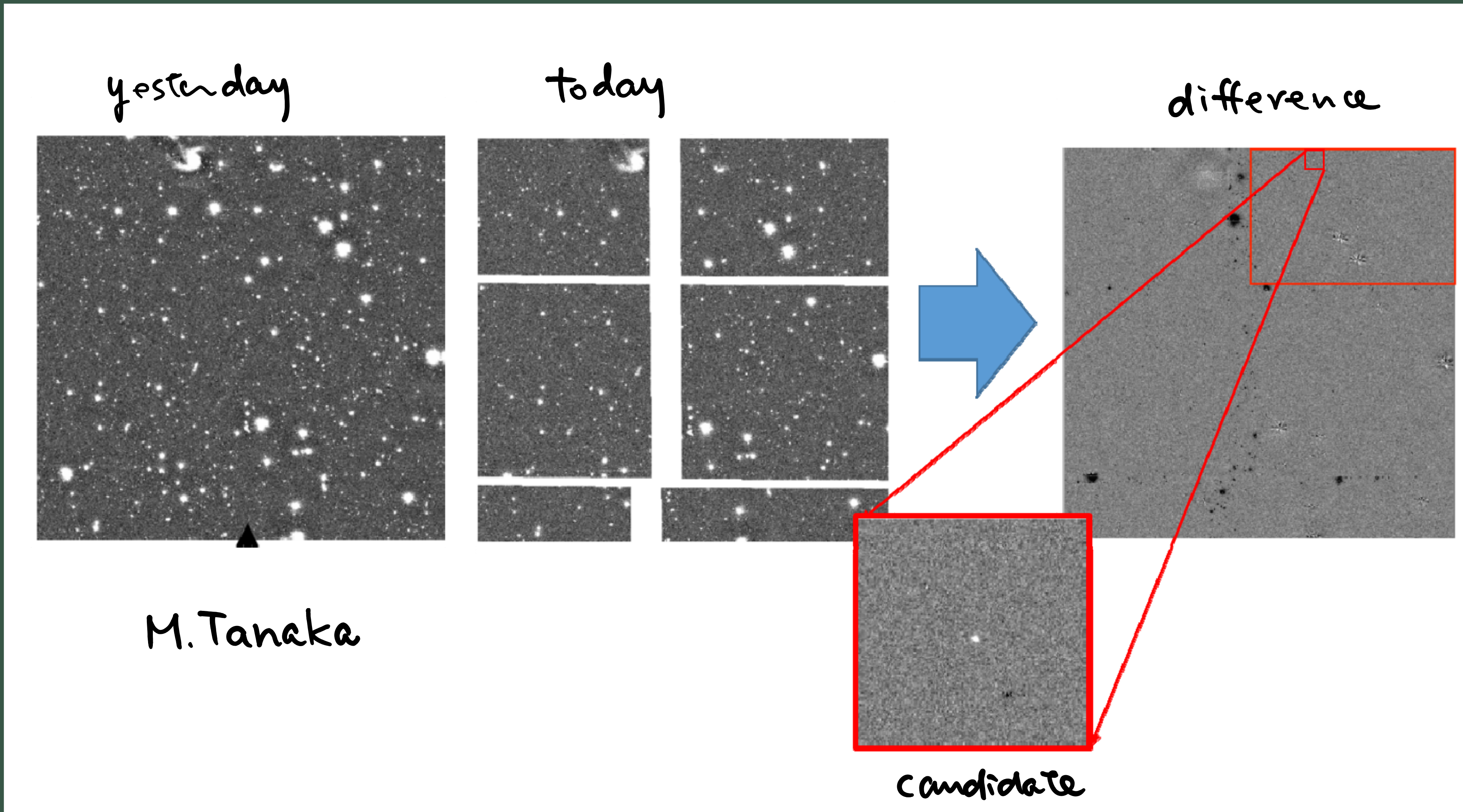
$\longrightarrow$ need to modify the model? hypothesis testing.

There are multiple methods for the estimation of 6 parameters

Type Ia SuperNovea & Cosmology



Q Finding Supernovea



more than 10,000 candidates. less than 100 supernovea.

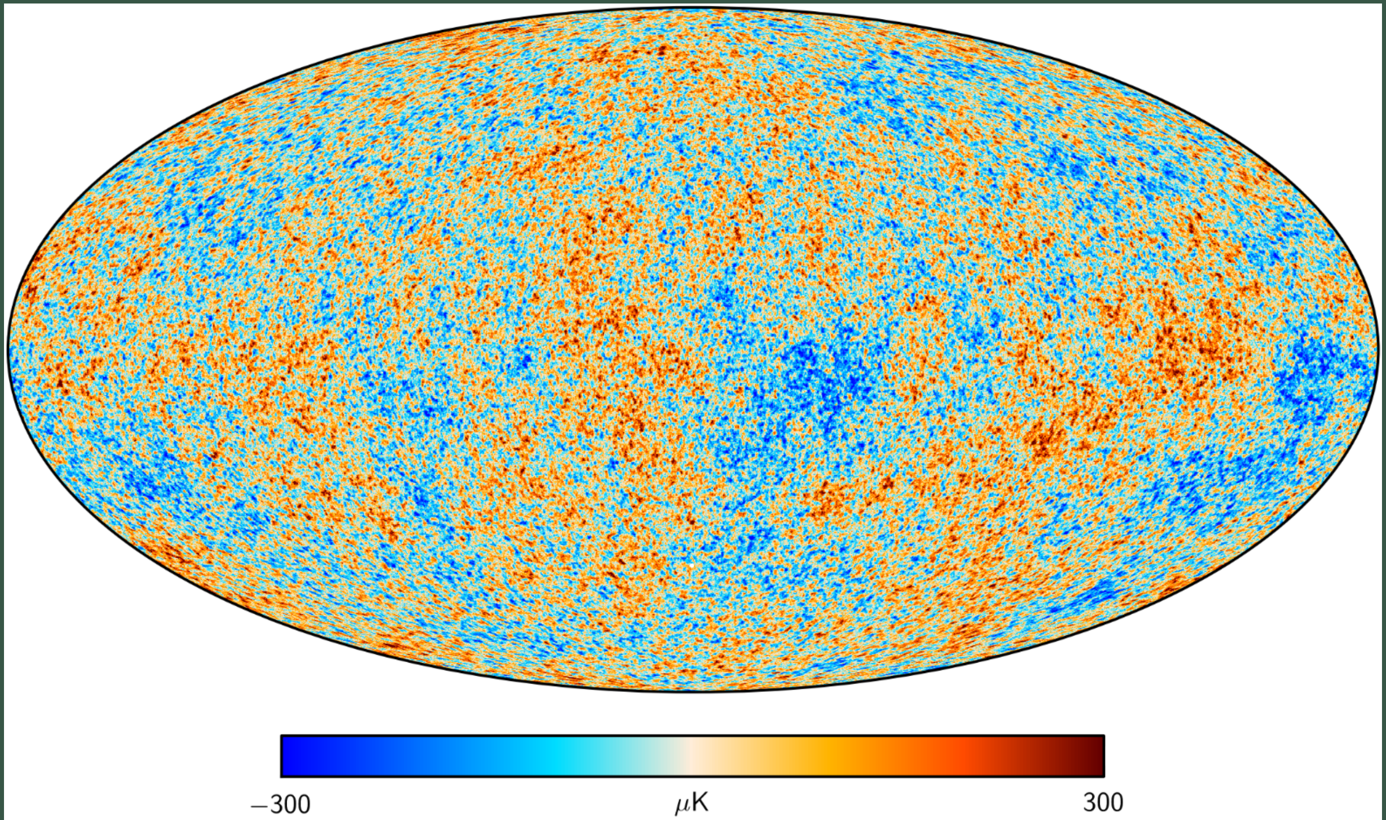
- Astro-statistics
  - Better measurement with Statistics

    EHT, Compton camera
  - Keep their scientific methods with Big-Data

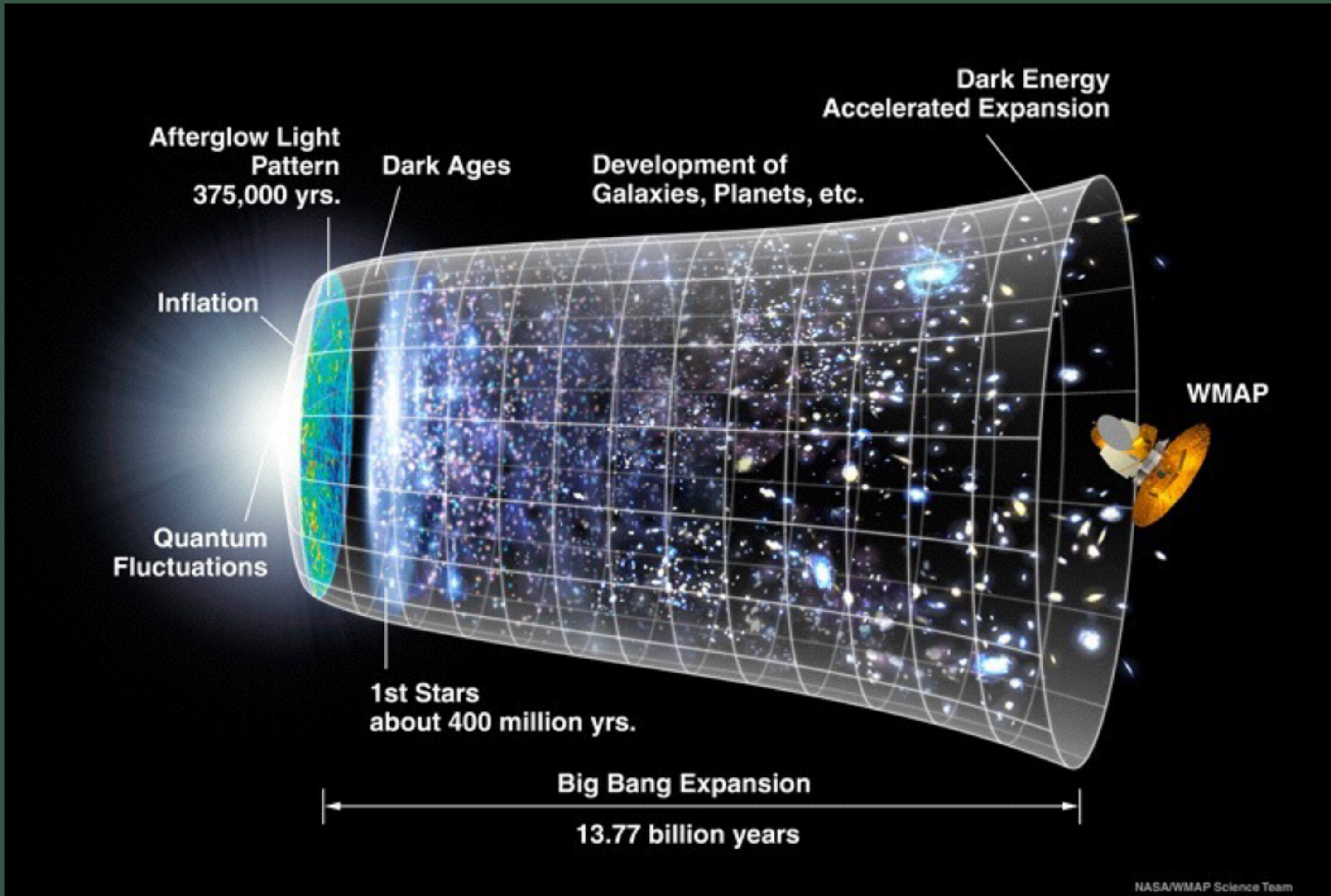
    HSC project
  - New method from statistics?

- Estimation

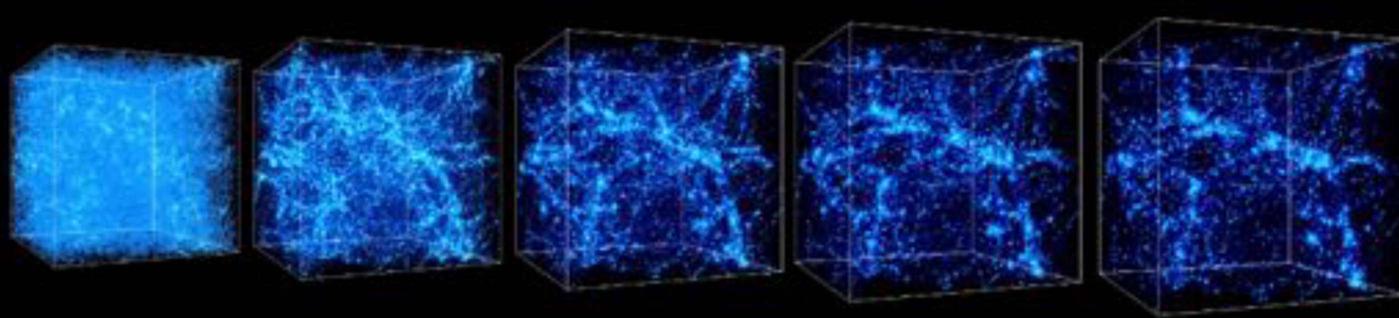
  - Weak lensing
  - Cosmological Microwave Background (CMB)



Planck : SMICA result



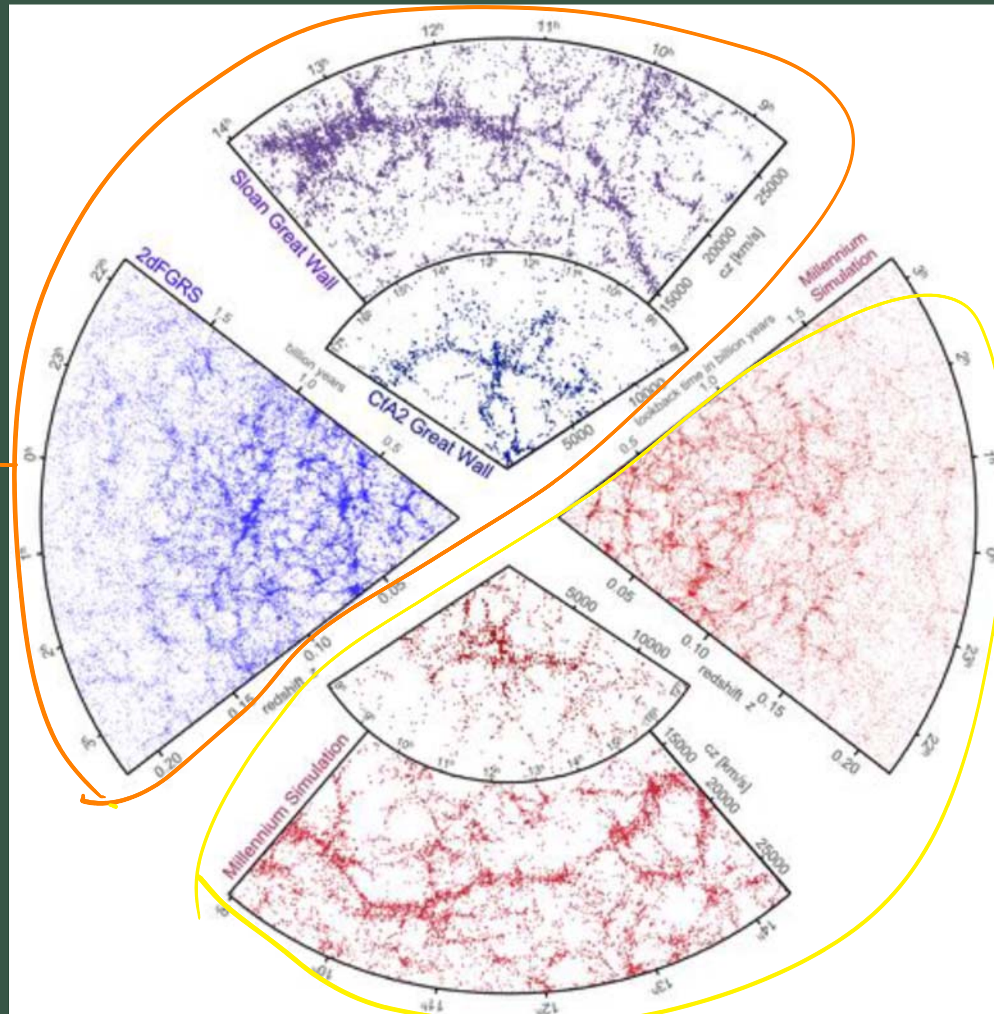
Model



Simulation

observation

T : statistics



Simulated universe

θ : parameters

what statistics T ?

how to estimate θ ?

Estimation of Cosmological Parameters

$$\text{Observations} \longrightarrow \hat{T}$$

$$\Updownarrow \text{ compare them}$$

$$\text{Simulations} \longrightarrow T(\theta)$$

$T(\theta)$: similar to ABC (Approximate Bayesian Computation)

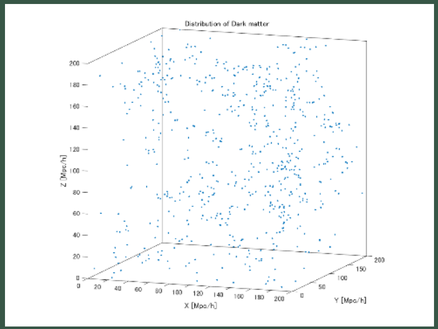
computational cost is high

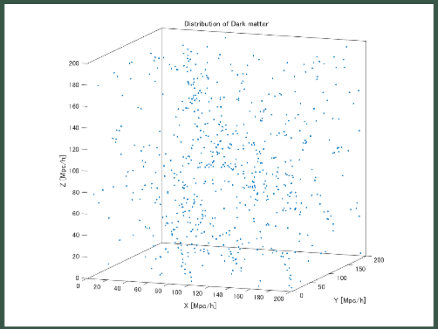
no analytical form

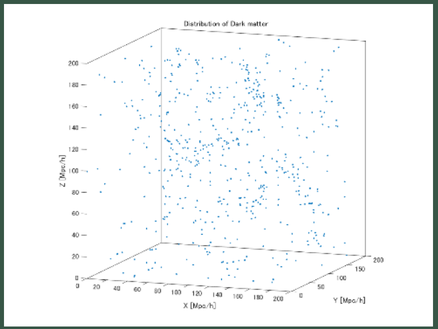
which $T(\theta)$ is good

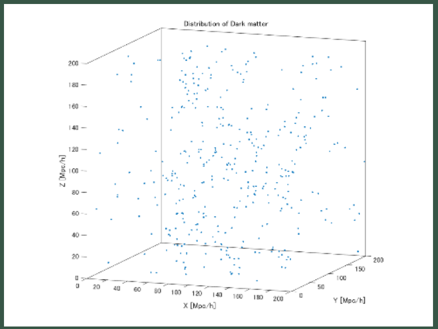
Simulation to Estimation

θ_1 θ_2 θ_3 θ_4









$T(\theta_1)$ $T(\theta_2)$ $T(\theta_3)$ $T(\theta_4)$

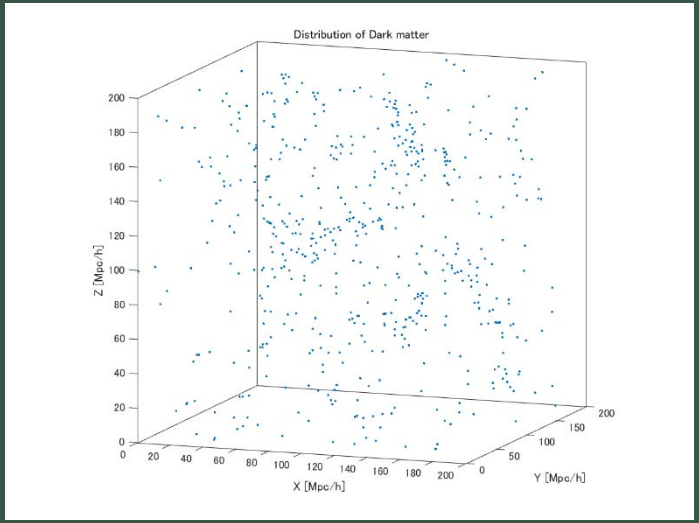
$p(T;\theta) \longrightarrow$ Fisher information $I(\theta)$

6 Statistics from Simulation

Commonly used statistics

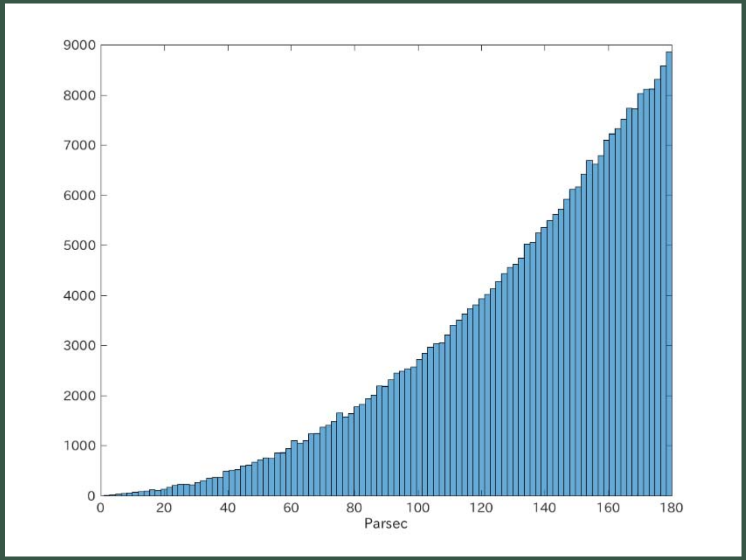
d_{ij} : distances

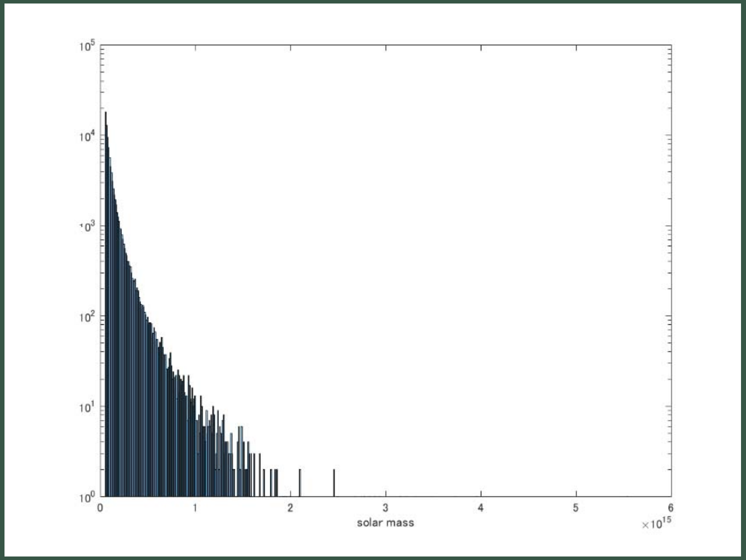
m_i : weight



$P(D;\theta)$

$P(m;\theta)$

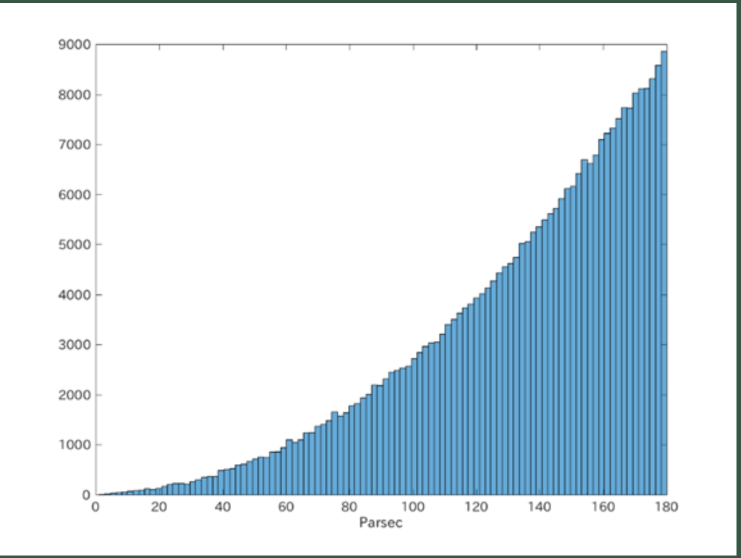




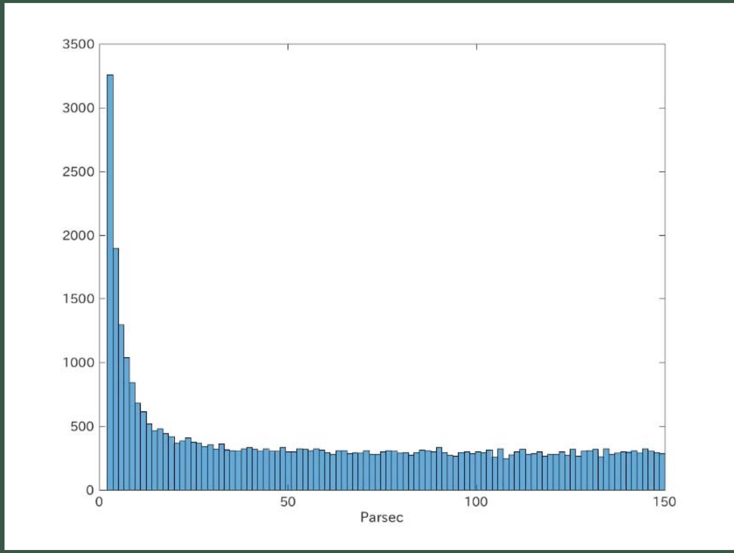
uniform : $p(D) \propto D^2$

Deviation from uniform dist. is important.

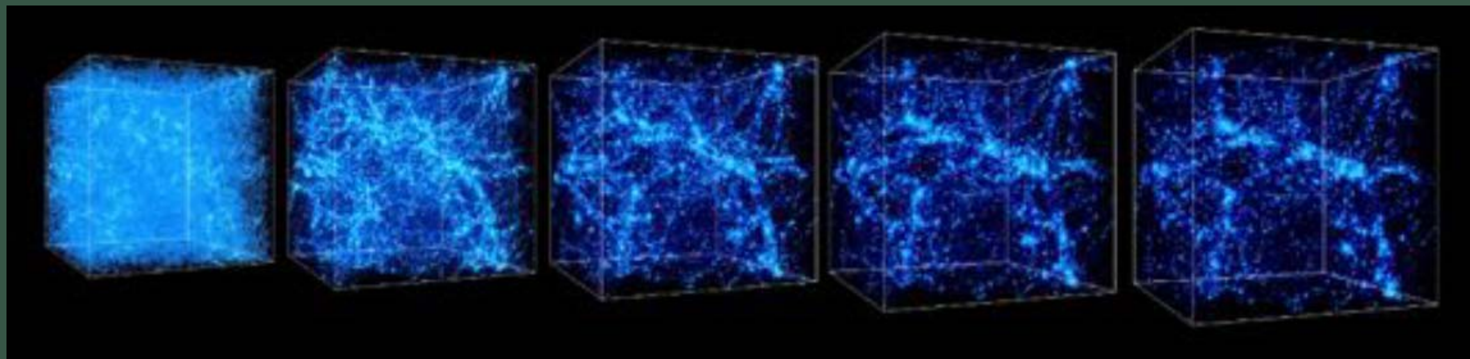
accept D_{ij} with a prob. $\propto \frac{1}{D_{ij}^2}$

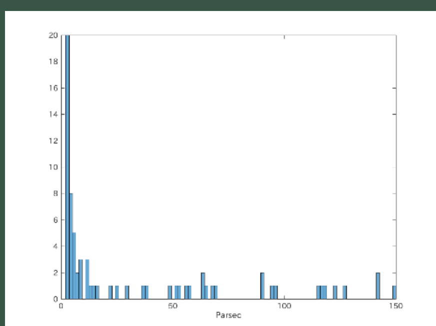


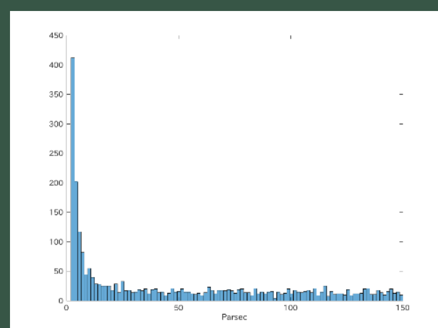
$\downarrow$

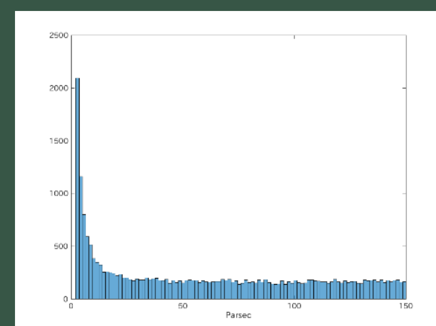


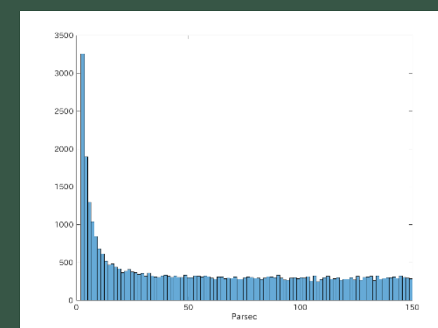
Simulated universe and Extracted Statistics











Fisher Information Matrix

- Samples from $p(T;\theta)$ are available $(10 \sim 100K)$
- Functional form is unknown.
- Have samples from $\theta_0, \theta_1, \cdots \theta_n$,

$$\Downarrow$$

$$D_\alpha(\theta;\theta+\delta) \simeq \frac{1}{2}\delta^T I(\theta)\delta$$

compute $I(\theta)$ approximately

K-nn method for computing $D_\alpha(\theta, \theta')$

$X_1, \cdots, X_N \sim p$, $Y_1, \cdots, Y_M \sim q$, $X, Y \in R^d$

$\rho_k(i)$: k-NN of X_i in $\{X_1, \cdots, X_N\}$

$\hat{\rho}_k(i)$: k-NN of X_i in $\{Y_1, \cdots, Y_M\}$

$$D_\beta(p : q) = \int \left(\frac{q}{p}\right)^{1-\beta} p\, dx \simeq \frac{1}{N} \sum_{i=1}^{N} \left(\frac{(N-1)\rho_k(i)}{M \hat{\rho}_k(i)} \right)^{1-\beta} B_{k,\beta}$$

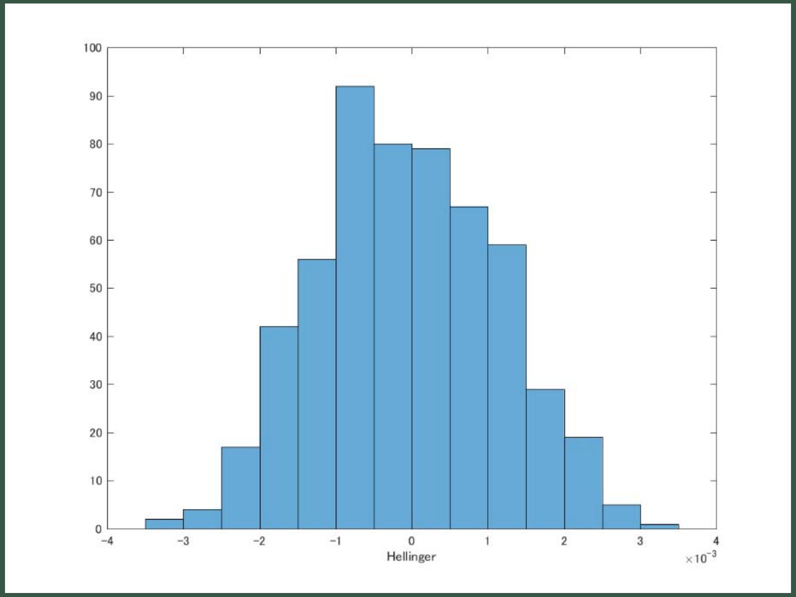
$$B_{k,\beta} = \frac{\Gamma(k)^2}{\Gamma(k-\beta+1)\Gamma(k+\beta-1)}$$

$$KL(p : q) = \frac{M}{N} \sum \log \frac{\hat{\rho}_k(i)}{\rho_k(i)} + \log \frac{M}{N-1}$$

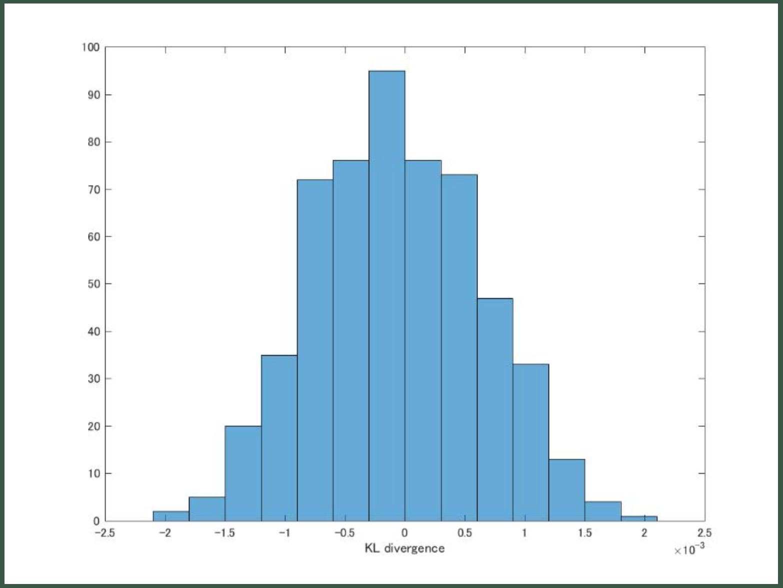
@ K-nn methods for computing $D_\alpha(\theta, \theta')$

$x \sim p(x; \theta_0)$: 24 samples.

$D_\alpha(\theta_0, \theta_0)$: $24 \times 23/2 = 276$



$D_0(\theta_0, \theta_0)$



$KL(\theta_0, \theta_0)$

@ Experiment Setup

$$D_\alpha(\theta_0 ; \theta+\delta_i) \simeq \frac{1}{2} \delta_i^T I(\theta_0) \delta_i$$

collecting these results for $\delta_1, \cdots \delta_L$, $\longrightarrow I(\theta_0)$

θ_0 : 24 samples

θ_i : $i = 1, \cdots, 40$

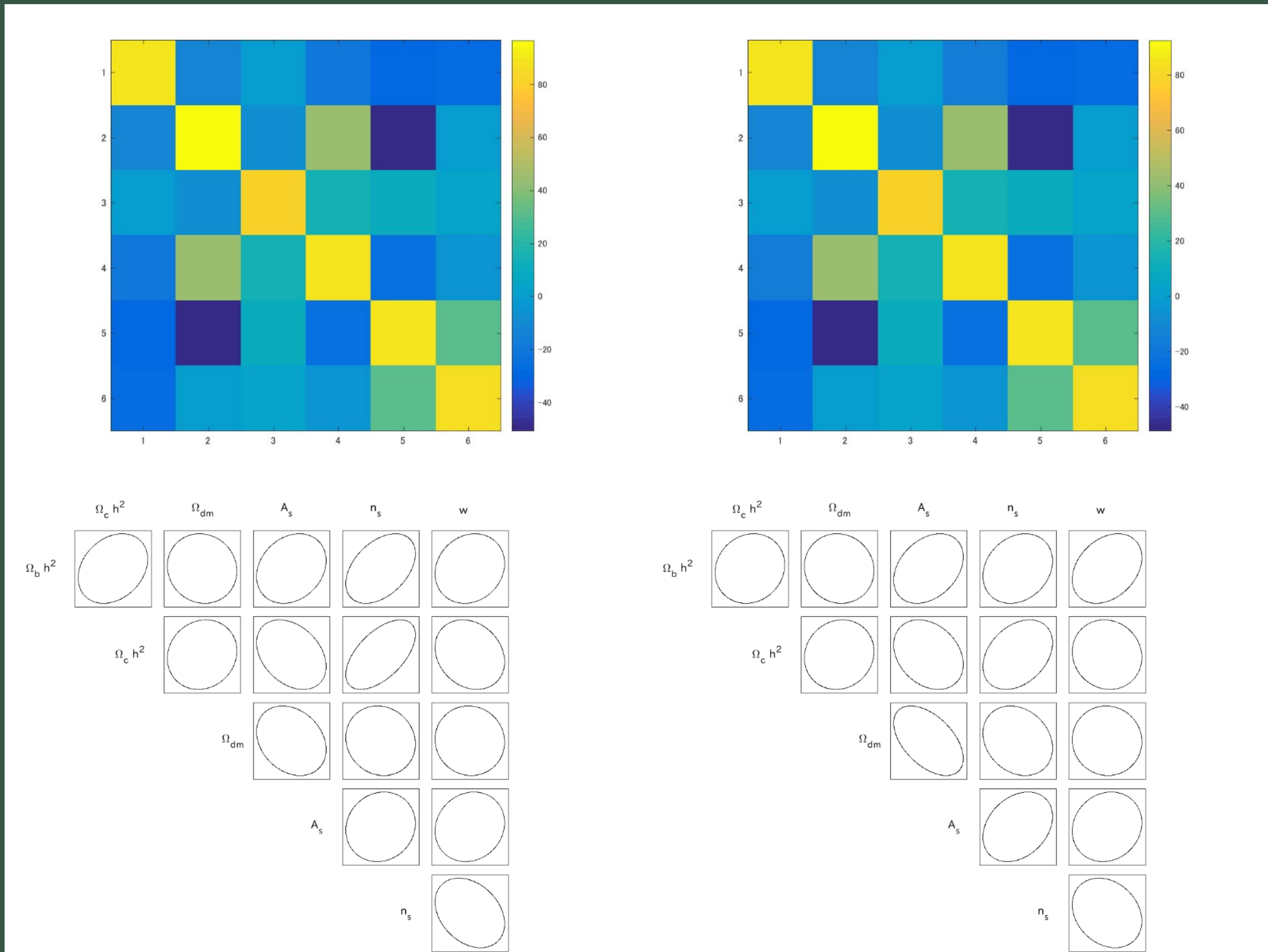
using these data set we computed Fisher matrix

for (2 points distance / mass distribution) with (Hellinger distance / K-L divergence)

2 points distance

$I(\theta)$

$I^{-1}(\theta)$



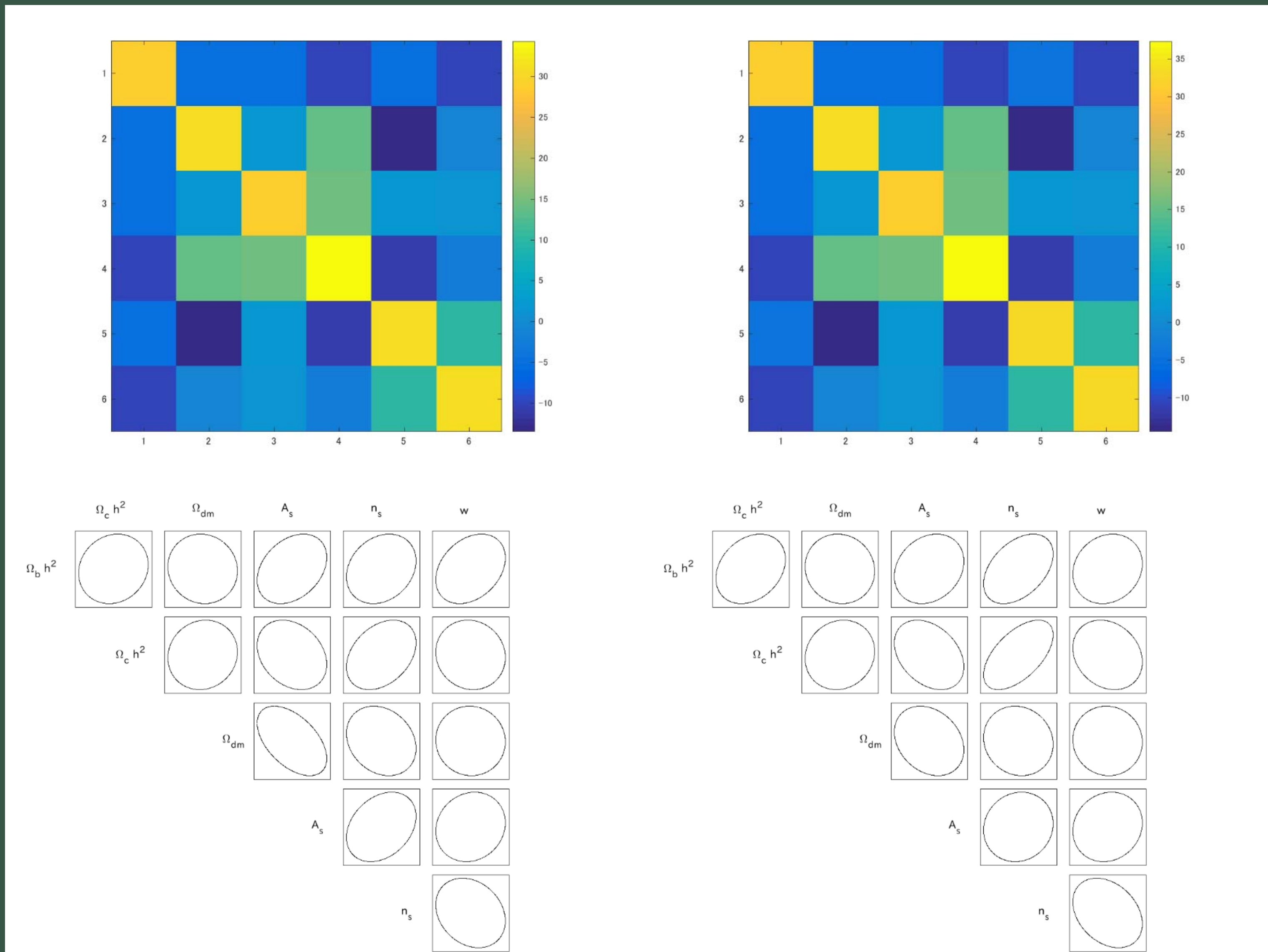
Hellinger distance

KL-divergence

mass

$I(\theta)$

$I(\theta)^{-1}$



Hellinger distance

KL divergence