Estimation in a Deformed Exponential Family

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Introduction

- Deformed exponential family is a generalized notion of an exponential family which was introduced by Naudts [1].
- Here we discuss about certain generalized notions of maximum likelihood estimator and estimation problem in a deformed exponential family.

Deformed Exponential Family

We formulate the deformed exponential family using a function F and we call it as F-exponential family.

Definition

Let $F : (0, \infty) \longrightarrow \mathbb{R}$ be any smooth function satisfying $F'(x) > 0$ and $F''(x) < 0$. Let Z be the inverse function of F. Define the standard form of an n-dimensional **F-exponential family** $\mathcal{S} = \{p(x; \theta)\}$ of probability distributions as

$$p(x;\theta) = Z\Big(\sum_{i=1}^{n} \theta^i x_i - \psi_F(\theta)\Big) \quad \text{or} \quad F(p(x;\theta)) = \sum_{i=1}^{n} \theta^i x_i - \psi_F(\theta) \quad (1)$$

where $x = (x_1, \cdots, x_n)$ is a set of random variables, $\theta = (\theta^1, \cdots, \theta^n)$ are the parameters and $\psi_F(\theta)$ is determined from the normalization condition. When $F(p) = \log p$ the F-exponential family is the exponential family.

Dually flat structures of the Deformed Exponential Family

The deformed exponential family has two dually flat structures, U-geometry by Naudts [1] and χ-geometry by Amari et al. [2].

U-geometry

Naudts [1] defined a divergence of Bregman type on a F-exponential family by

$$D^F(p,q) = \int \left(\int_q^p (F(u) - F(q))du \right) dx \quad (2)$$

This divergence induces a dually flat structure $(g^{D^F}, \nabla^{D^F}, \nabla^{D^{*F}})$ called the U-geometry.

χ-geometry

Amari et al. [2] defined a divergence D_F on the F-exponential family as

$$D_F(p,r) = \frac{1}{h_F(\theta_1)} \int (F(p) - F(r)) \frac{1}{F'(p)} dx \quad (3)$$

where $h_F(\theta) = \int \frac{1}{F'(p)} dx$.

D_F induces a dually flat structure $(g^{D_F}, \nabla^{D_F}, \nabla^{D_F^*})$ which is called the χ-geometry.

U-estimator

Let $\mathbf{x}_N = (x^1, \cdots, x^N)$ be N independent observations from $p(x;\theta) \in \mathcal{S}$. Eguchi et al. [3] defined the U-estimator $\hat{\theta}_U$ using an increasing convex function U by

$$\hat{\theta}_U = \arg\min_{\theta \in \mathbb{E}} L_U(\theta) \quad (4)$$

where $L_U(\theta)$ is the U-loss function defined as

$$L_U(\theta) = -\frac{1}{N} \sum_{i=1}^{N} \xi(p(x^i;\theta)) + \int U(\xi(p(x;\theta)))dx \; ; \; (U^*)'(t) = \xi(t)$$

Generalized Cramer-Rao Lower Bound

The U-estimator is not optimal with respect to the Cramer-Rao lower bound. Naudts [1] defined a generalized Cramer-rao bound. We give a proof of the generalized Cramer-Rao bound defined by Naudts using a generalized score vector and an F-escort probability density function.

Define **F-expectation** and **F-variance** of a random variable $X \sim p(x;\theta)$ as follows

$$E_{\hat{p}_F}[X] = \frac{1}{h_F(\theta)} \int x \frac{1}{F'(p)} dx \quad (5)$$

$$\mathrm{Var}_F(X) = E_{\hat{p}_F}[(X - E_{\hat{p}_F}(X))^2] \quad (6)$$

Theorem

Let X be a random variable with density $p(x;\theta) \in \mathcal{S}$. Let $T = t(X)$ be an unbiased estimator for $\psi(\theta)$ so that $E_p[t(X)] = \psi(\theta)$. Also let the F-expectation of $t(X)$ is $E_{\hat{p}_F}[t(X)] = \phi(\theta)$. Then the F-variance satisfies the lower bound

$$\mathrm{Var}_F(T) \geq \frac{|\,\psi'(\theta)\,|^2}{g^N(\theta)} \quad (7)$$

where $g^N(\theta) = h_F(\theta) g^G(\theta)$, g^G is the G-metric with $G(p) = pF'(p)$.

Theorem

Let $\mathcal{S} = \{p(x;\theta) = Z(\theta x - \psi_F(\theta))\}$ be a F-exponential family and let $\eta = E_p[x]$ be the dual coordinate in the U-geometry. Then U-estimator $\hat{\eta}_U = \bar{x}$ for η is optimal with respect to the generalized Cramer-Rao bound defined by Naudts. That is,

$$\mathrm{Var}_F(\hat{\eta}_U) = \frac{1}{g^N(\eta)}. \quad (8)$$

Conclusion

- A proof of the generalized Cramer-Rao bound defined by Naudts [1] is given.
- Then we give a proof of the result that in a deformed exponential family the U-estimator for the dual coordinate in the U-geometry is optimal with respect to the generalized Cramer-Rao bound defined by Naudts.

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