ON ROBERTSON-TYPE UNCERTAINTY PRINCIPLES

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A generalisation of the classical covariance for quantum mechanical observables has previously been presented by Gibilisco, Hiai and Petz. Gibilisco and Isola has proved that the usual quantum covariance gives the sharpest inequalities for the determinants of covariance matrices. We introduce new generalisations of the classical covariance which gives better inequalities, furthermore it has a direct geometric interpretation.

Keywords: uncertainty principle, quantum Fisher information

In quantum mechanics the notion of the (symmetrized) covariance of the observables A and B at a given state D was defined as

$$\mathrm{Cov}_D(A,B) = \frac{1}{2}\left(\mathrm{Tr}(DAB) + \mathrm{Tr}(DBA)\right) - \mathrm{Tr}(DA)\,\mathrm{Tr}(DB)$$

and the variance as $\mathrm{Var}_D(A) = \mathrm{Cov}_D(A,A)$. In 1930 Scrödinger proved the uncertainty relation

$$\mathrm{Var}_D(A)\,\mathrm{Var}_D(B) - \mathrm{Cov}_D(A,B)^2 \geq \frac{1}{4}\left|\mathrm{Tr}(D\,[A,B])\right|^2$$

which was generalized by Robertson in 1934 [6] for the set of observables $(A_i)_{1,\dots,N}$ as

$$\det\left(\left[\mathrm{Cov}_D(A_h,A_j)\right]_{h,j=1,\dots,N}\right) \geq \det\left(\left[-\frac{\mathrm{i}}{2}\,\mathrm{Tr}(D\,[A_h,A_j])\right]_{h,j=1,\dots,N}\right).$$

Later several notions of covariance occured, such as the quantum f-covariance [5]

$$\mathrm{Cov}_D^f(A,B) = \mathrm{Tr}\left(Af(L_{n,D}R_{n,D}^{-1})R_{n,D}(B)\right),$$

the antisymmetric f-covariance and the symmetric f-covariance

$$\mathrm{qCov}^{as}_{D,f}(A,B) = \frac{f(0)}{2} \langle \mathrm{i}\,[D,A]\,, \mathrm{i}\,[D,B] \rangle_{D,f}$$
$$\mathrm{qCov}^{s}_{D,f}(A,B) = \frac{f(0)}{2} \langle \{D,A\}\,, \{D,B\} \rangle_{D,f}\,,$$

where $[.,.]$ is the commutator, $\{.,.\}$ the anticommutator and the scalar product

$$\langle A,B \rangle_{D,f} = \mathrm{Tr}\left(A \left(R_D^{1/2} f(L_D R_D) R_D^{1/2} \right)^{-1} (B) \right)$$

is induced by an operator monotone function f, according to Petz classification theorem [4]. Gibilisco and Isola in 2006 conjectured that

$$\det(\mathrm{Cov}_D) \geq \det(\mathrm{qCov}^{as}_{D,f})$$

holds [3]. The conjecture was proved by Andai [1] and Gibilisco, Imparato and Isola [2] in 2008. We show the more accurate inequality

$$\det(\mathrm{Cov}_D) \geq \det(\mathrm{qCov}^{s}_{D,f}) \geq \det(\mathrm{qCov}^{as}_{D,f})$$

and the following estimation for the gap between the symmetric and antisymmetric covariance

$$\det(\mathrm{qCov}^{s}_{D,f}) - \det(\mathrm{qCov}^{as}_{D,f}) \geq (2f(0))^N \det(\mathrm{Cov}^{f_{RLD}}_D),$$

where $f_{RLD}(x) = \frac{2x}{1+x}$. Moreover we show that the symmetric covariance generated by the function $f_{opt} = \frac{1}{2}\left(\frac{1+x}{2} + \frac{2x}{1+x}\right)$ is universal in the following sense. For every function g the inequality $\det(\mathrm{qCov}^{s}_{D,f_{opt}}) \geq \det(\mathrm{qCov}^{as}_{D,g})$ holds.

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